

A KOWALSKI-SŁODKOWSKI THEOREM FOR 2-LOCAL *-HOMOMORPHISMS ON VON NEUMANN ALGEBRAS

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ABSTRACT. It is established that every (not necessarily linear) 2-local *-homomorphism from a von Neumann algebra into a C^* -algebra is linear and a *-homomorphism. In the setting of (not necessarily linear) 2-local *-homomorphism from a compact C^* -algebra we prove that the same conclusion remains valid. We also prove that every 2-local Jordan *-homomorphism from a JBW^* -algebra into a JB^* -algebra is linear and a Jordan *-homomorphism.

1. INTRODUCTION

The Gleason-Kahane-Żelazko theorem (cf. [15, 21]), a fundamental contribution in the theory of Banach algebras, asserts that every unital linear functional F on a complex unital Banach algebra A such that, $F(a)$ belongs to the spectrum, $\sigma(a)$, of a for every $a \in A$, is multiplicative. In modern terminology, this is equivalent to say that every unital linear local homomorphism from a unital complex Banach algebra A into \mathbb{C} is multiplicative. We recall that a linear mapping T from a Banach algebra A into a Banach algebra B is said to be a *local homomorphism* if for every a in A there exists a homomorphism $\Phi_a : A \rightarrow B$, depending on a , satisfying $T(a) = \Phi_a(a)$. *Local derivations* are similarly defined. Kadison [20] and Larson and Sourour [26] made the first contributions to the theory of local derivations and local automorphisms on Banach algebras, respectively. Briefly speaking, Johnson [19], culminated the studies on local derivations, showing that every local derivation from a C^* -algebra A into a Banach A -bimodule is a derivation. A wide list of authors, studied local homomorphisms between C^* -algebras (we refer to the introduction of [30] for a recent expository paper on linear local homomorphisms).

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After the Gleason-Kahane-Żelazko theorem was established, Kowalski and Ślodkowski [25] showed that at the cost of requiring the local behavior at two points, the condition of linearity can be dropped, that is, suppose A is a complex Banach algebra (not necessarily commutative nor unital), then every (not necessarily linear) mapping $T : A \rightarrow \mathbb{C}$ satisfying $T(0) = 0$ and $T(x - y) \in \sigma(x - y)$, for every $x, y \in A$, is multiplicative and linear.

Let A and B be two (complex) Banach algebras. Following the standard notation introduced by P. Šemrl in [33], a (not necessarily linear nor continuous) mapping $T : A \rightarrow B$ is said to be a *2-local homomorphism* (respectively, a *2-local isomorphism*) if for every $a, b \in A$ there exists a bounded (linear) homomorphism (respectively, a bounded isomorphism) $\Phi_{a,b} : A \rightarrow B$, depending on a and b , such that $\Phi_{a,b}(a) = T(a)$ and $\Phi_{a,b}(b) = T(b)$. *2-local Jordan homomorphisms*, *2-local Jordan monomorphisms* and *2-local Jordan automorphisms* are defined in a similar fashion. We recall that a linear mapping $\Phi : A \rightarrow B$ is said to be a Jordan homomorphism whenever $\Phi(a^2) = \Phi(a)^2$ (equivalently, Φ preserves the Jordan products of the form $a \circ b := \frac{1}{2}(ab + ba)$).

When A and B are C^* -algebras, a mapping $T : A \rightarrow B$ is called a *2-local $*$ -homomorphism* (respectively, a *2-local $*$ -isomorphism*) if for every $a, b \in A$ there exists a $*$ -homomorphism (respectively, a $*$ -isomorphism) $\Phi_{a,b} : A \rightarrow B$, depending on a and b , such that $\Phi_{a,b}(a) = T(a)$ and $\Phi_{a,b}(b) = T(b)$. In the case $A = B$, 2-local isomorphisms, 2-local Jordan isomorphisms, 2-local $*$ -isomorphisms, and 2-local Jordan $*$ -isomorphisms are called *2-local automorphisms*, *2-local Jordan automorphisms*, *2-local $*$ -automorphisms*, and *2-local Jordan $*$ -automorphisms*, respectively.

According to this more recent notation, the result established by Kowalski and Ślodkowski in [25] proves that every (not necessarily linear) 2-local homomorphism T from a (not necessarily commutative nor unital) complex Banach algebra A into the complex field \mathbb{C} is linear and multiplicative. Consequently, every (not necessarily linear) 2-local homomorphism T from A into a commutative C^* -algebra is linear and multiplicative.

In 1997, Šemrl [33] proves that for every infinite-dimensional separable Hilbert space H , every 2-local automorphism $T : B(H) \rightarrow B(H)$ is an automorphism. Short and elegant proofs of Šemrl's theorem for 2-local automorphisms of matrix algebras were obtained by Molnár [29] and Kim and Kim [23]. In the just quoted paper [29], Molnár also proves that 2-local automorphisms of operator algebras containing all compact operators on a Banach space with a Schauder basis are automorphisms. More studies on 2-local Jordan automorphisms on the algebra of all $n \times n$ real or complex matrices were developed by Fošner in [12], where it is additionally established that every 2-local Jordan automorphism of any subalgebra of $B(X)$ which contains the ideal of all compact operators on X , where X is a real or complex separable Banach space is either an automorphism or an anti-automorphism.

The same author investigates, in [13], 2-local $*$ -automorphisms, 2-local $*$ -antiautomorphisms, and 2-local Jordan $*$ -derivations on $M_n(\mathbb{C})$, $B(H)$ and certain unital standard operator algebras on H with involution.

In 2012, Ayupov and Kudaybergenov introduce new techniques to generalize Šemrl's theorem for arbitrary Hilbert spaces, showing that 2-local automorphisms on the algebra $B(H)$ on an arbitrary (no separability is assumed) Hilbert space H are automorphisms (see [2]).

Assuming linearity Hadwin and Li [17, Theorem 3.7] prove that every bounded linear and unital 2-local homomorphism (respectively, 2-local $*$ -homomorphism) from a unital C^* -algebra of real rank zero into itself is a homomorphism (respectively, a $*$ -homomorphism). Under the same additional assumptions, Pop [31] establishes that every bounded linear 2-local homomorphism (respectively, 2-local $*$ -homomorphism) from a von Neumann algebra into another C^* -algebra is a homomorphism (respectively, a $*$ -homomorphism) [31, Corollary 3.6]. In 2006, Liu and Wong prove that every linear 2-local automorphism T of a C^* -algebra whose range is a C^* -algebra is an algebra homomorphism. Actually every bounded linear 2-local homomorphism between C^* -algebras is a homomorphism (cf. [30]).

In the setting of C^* -algebras, Kim and Kim [24] prove that every surjective 2-local $*$ -automorphism on a prime C^* -algebra or on a C^* -algebra such that the identity element is properly infinite is a $*$ -automorphism. An illustrative example provided by Györy in [16] proves the existence of non-surjective linear 2-local $*$ -automorphisms between $C_0(L)$ -spaces, which shows that the hypothesis concerning surjectivity cannot be relaxed in the result established by Kim and Kim.

In this paper we study (not necessarily linear nor continuous) 2-local homomorphisms and 2-local $*$ -homomorphisms between general C^* -algebras. Our main result (Theorem 2.12) establishes that every (not necessarily linear) 2-local $*$ -homomorphism from a von Neumann algebra into a C^* -algebra is linear and a $*$ -homomorphism. The techniques involve several applications of the Bunce-Wright-Mackey-Gleason theorem [6, 7], and a subtle variant for dual or compact C^* -algebras due to Aarnes [1]. In the setting of (not necessarily linear) 2-local $*$ -homomorphism from a compact C^* -algebra we prove that the conclusion of Theorem 2.12 remains valid (Theorem 3.3).

Finally, in section 4, we prove that every (not necessarily linear) 2-local Jordan $*$ -homomorphism from a JBW^* -algebra into a JB^* -algebra is linear and a Jordan $*$ -homomorphism (compare Theorem 4.5).

It would be of great interest to extend Theorem 2.12 (respectively, Theorem 4.5) to C^* -algebras (respectively, JB^* -algebras), but in this case projections and partial isometries are useless, because there exist C^* -algebras lacking of non-trivial projections.

2. 2-LOCAL *-HOMOMORPHISMS ON VON NEUMANN ALGEBRAS

When A is a C^* -algebra or a JB^* -algebra, the symbol A_{sa} will stand for the set of all self-adjoint elements in A .

We begin our study gathering some basic properties on 2-local mappings. A mapping f between Banach algebras A and B is said to be *zero products preserving* if the implication

$$ab = 0 \Rightarrow f(a)f(b) = 0$$

holds for every $a, b \in A$. Let us recall that elements a, b in a C^* -algebra A are said to be *orthogonal* (denoted by $a \perp b$) whenever $ab^* = b^*a = 0$. A map f from A to another C^* -algebra is called *orthogonality preserving* when $f(a) \perp f(b)$, for every $a, b \in A$ with $a \perp b$.

Throughout this note, given a 2-local homomorphism (respectively, a 2-local $*$ -homomorphism) $T : A \rightarrow B$ between Banach algebras (respectively, between C^* -algebras) and elements $a, b \in A$, the symbol $\Phi_{a,b}$ will denote a (linear) homomorphism (respectively, $*$ -homomorphism) satisfying $T(a) = \Phi_{a,b}(a)$ and $T(b) = \Phi_{a,b}(b)$.

Lemma 2.1. *Let $T : A \rightarrow B$ be a (not necessarily linear nor continuous) 2-local homomorphism between (complex) Banach algebras. The following statements hold:*

- (a) T is 1-homogeneous, that is, $T(\lambda a) = \lambda T(a)$ for every $a \in A$, $\lambda \in \mathbb{C}$;
- (b) T maps idempotents in A to idempotents in B ;
- (c) T is zero-products preserving;
- (d) $T(a)^2 = T(a^2)$, for every $a \in A$.

Under the additional hypothesis of T being a 2-local $$ -homomorphism between C^* -algebras we have:*

- (e) T is 1-Lipschitzian and automatically continuous, that is,

$$\|T(a) - T(b)\| \leq \|a - b\|,$$

for every $a, b \in A$;

- (f) $T(a^*) = T(a)^*$, for every $a \in A$;
- (g) T maps projections in A to projections in B ;
- (h) T preserves orthogonality.
- (i) $T(a)T(a)^* = T(aa^*)$, and $T(a)^*T(a) = T(a^*a)$, for every $a \in A$.

Proof. (a) For each $a \in A$, $\lambda \in \mathbb{C}$, let us consider the homomorphism $\Phi_{a,\lambda a}$. Then $T(\lambda a) = \Phi_{a,\lambda a}(\lambda a) = \lambda \Phi_{a,\lambda a}(a) = \lambda T(a)$.

(b) For each idempotent e in A , we have

$$T(e)^2 = \Phi_{e,e}(e)^2 = \Phi_{e,e}(e^2) = \Phi_{e,e}(e) = T(e).$$

(c) Suppose $ab = 0$ for certain $a, b \in A$. Then

$$T(a)T(b) = \Phi_{a,b}(a)\Phi_{a,b}(b) = \Phi_{a,b}(ab) = 0.$$

(d) Pick an element $a \in A$ and consider the homomorphism Φ_{a,a^2} . Then

$$T(a)^2 = \Phi_{a,a^2}(a)^2 = \Phi_{a,a^2}(a^2) = T(a^2).$$

(e) Having in mind that every $*$ -homomorphism between C^* -algebras is contractive, given $a, b \in A$, the $*$ -homomorphism $\Phi_{a,b}$ can be applied to prove the inequality

$$\|T(a) - T(b)\| = \|\Phi_{a,b}(a) - \Phi_{a,b}(b)\| = \|\Phi_{a,b}(a - b)\| \leq \|a - b\|,$$

which gives the desired statement.

(f) For each a in A , we have

$$T(a)^* = \Phi_{a,a^*}(a)^* = \Phi_{a,a^*}(a^*) = T(a^*).$$

The statements (g) and (h) are clear from the previous ones. Finally, to prove (i), we take a $*$ -homomorphism Φ_{a,aa^*} satisfying $T(a) = \Phi_{a,aa^*}(a)$ and $T(aa^*) = \Phi_{a,aa^*}(aa^*)$, and we observe that

$$T(a)T(a)^* = \Phi_{a,aa^*}(a)\Phi_{a,aa^*}(a)^* = \Phi_{a,aa^*}(aa^*) = T(aa^*).$$

□

The next technical lemma establishes that every 2-local homomorphism between Banach algebras is additive on a couple of idempotents whose products are zero.

Lemma 2.2. *Let $T : A \rightarrow B$ be a (not necessarily linear nor continuous) 2-local homomorphism between (complex) Banach algebras. Let e and f be two idempotents in A satisfying $ef = fe = 0$. Then $T(e + f) = T(e) + T(f)$.*

Proof. The identity $T(e) + T(f - e) = \Phi_{e,f-e}(e) + \Phi_{e,f-e}(f - e) = \Phi_{e,f-e}(e + (f - e)) = \Phi_{e,f-e}(f)$, implies that $T(e) + T(f - e)$ is an idempotent in B . Therefore, by Lemma 2.1,

$$\begin{aligned} T(e) + T(f - e) &= (T(e) + T(f - e))^2 \\ &= T(e)^2 + T(f - e)^2 + T(e)T(f - e) + T(f - e)T(e) \\ &= T(e) + T((f - e)^2) + \Phi_{e,f-e}(e(f - e)) + \Phi_{e,f-e}((f - e)e) \\ &= T(e) + T(f + e) - T(e) - T(e), \end{aligned}$$

which gives

$$2T(e) = T(e + f) + T(e - f).$$

Replacing e with f we get

$$2T(f) = T(e + f) + T(f - e) = T(e + f) - T(e - f),$$

and hence $T(e) + T(f) = T(e + f)$. □

We shall establish now the linearity of every 2-local homomorphism on a finite linear combination of idempotents having zero products.

Lemma 2.3. *Let $T : A \rightarrow B$ be a (not necessarily linear nor continuous) 2-local homomorphism between (complex) Banach algebras. Let e_1, \dots, e_n be idempotents in A satisfying $e_i e_j = e_j e_i = 0$ for every $i \neq j$. Then*

- (a) $T\left(\sum_{i=1}^n e_i\right) = \sum_{i=1}^n T(e_i);$
- (b) $T\left(\sum_{i=1}^n \lambda_i e_i\right) = \sum_{i=1}^n \lambda_i T(e_i),$ for every $\lambda_1, \dots, \lambda_n \in \mathbb{C}.$

Proof. We shall prove (a) by induction on n . The case $n = 1$ is clear, while the case $n = 2$ is established in Lemma 2.2. Let e_1, \dots, e_n, e_{n+1} be idempotents in A satisfying $e_i e_j = e_j e_i = 0$ for every $i \neq j$. The elements $e = e_1 + \dots + e_n$ and e_{n+1} are idempotents in A satisfying $e e_{n+1} = e_{n+1} e = 0$. Lemma 2.2 combined with the induction hypothesis assure that

$$T\left(\sum_{i=1}^{n+1} e_i\right) = T(e + e_{n+1}) = T(e) + T(e_{n+1}) = \sum_{i=1}^n T(e_i) + T(e_{n+1}).$$

(b) As in the proof of Lemma 2.2, $\Phi_{a,b}$ will denote a (linear) homomorphism satisfying $T(a) = \Phi_{a,b}(a)$ and $T(b) = \Phi_{a,b}(b)$. Fix $j \in \{1, \dots, n\}$ and set $z = \sum_{i=1}^n \lambda_i e_i$. We first calculate

$$\begin{aligned} (2.1) \quad T\left(\sum_{i=1}^n \lambda_i e_i\right) T(e_j) &= \Phi_{z,e_j}\left(\sum_{i=1}^n \lambda_i e_i\right) \Phi_{z,e_j}(e_j) \\ &= \Phi_{z,e_j}\left(\left(\sum_{i=1}^n \lambda_i e_i\right) e_j\right) = \Phi_{z,e_j}(\lambda_j e_j) = \lambda_j T(e_j). \end{aligned}$$

To simplify notation, we write $e = \sum_{i=1}^n e_i$. In this case

$$\begin{aligned} T\left(\sum_{i=1}^n \lambda_i e_i\right) &= \Phi_{z,e}\left(\sum_{i=1}^n \lambda_i e_i\right) = \Phi_{z,e}\left(\left(\sum_{i=1}^n \lambda_i e_i\right) e\right) \\ &= \Phi_{z,e}\left(\sum_{i=1}^n \lambda_i e_i\right) \Phi_{z,e}(e) = T\left(\sum_{i=1}^n \lambda_i e_i\right) T(e) = T\left(\sum_{i=1}^n \lambda_i e_i\right) T\left(\sum_{j=1}^n e_j\right) \\ &= (\text{by (a)}) = T\left(\sum_{i=1}^n \lambda_i e_i\right) \sum_{j=1}^n T(e_j) = (\text{by (2.1)}) = \sum_{j=1}^n \lambda_j T(e_j). \end{aligned}$$

□

Lemma 2.4. *Let $T : A \rightarrow B$ be a (not necessarily linear) 2-local $*$ -homomorphism between C^* -algebras. Then $T(a + ib) = T(a) + iT(b)$, for every $a, b \in A_{sa}$.*

Proof. Given $a, b \in A_{sa}$, we consider the $*$ -homomorphisms $\Phi_{a,a+ib}$ and $\Phi_{b,a+ib}$. The identities

$$T(a + ib) = \Phi_{a,a+ib}(a) + i\Phi_{a,a+ib}(b) = T(a) + i\Phi_{a,a+ib}(b),$$

and

$$T(a + ib) = \Phi_{b,a+ib}(a) + i\Phi_{b,a+ib}(b) = \Phi_{b,a+ib}(a) + iT(b),$$

together with Lemma 2.1(f), imply that

$$T(a + ib) + T(a - ib) = T(a + ib) + T(a + ib)^* = 2T(a),$$

and

$$2iT(b) = T(a + ib) - T(a - ib),$$

which prove $T(a + ib) = T(a) + iT(b)$. \square

Theorem 2.5. *Let \mathcal{M} be a von Neumann algebra with no Type I_2 direct summand and let B be a C^* -algebra. Suppose $T : \mathcal{M} \rightarrow B$ is a (not necessarily linear) 2-local $*$ -homomorphism. Then T is linear and a $*$ -homomorphism.*

Proof. By Lemma 2.1(e), T is 1-Lipschitzian and hence automatically continuous. Let $\mathcal{P}(\mathcal{M})$ denote the lattice of projections of \mathcal{M} and define a mapping $\mu : \mathcal{P}(\mathcal{M}) \rightarrow B$ by $\mu(p) = T(p)$, for every $p \in \mathcal{P}(\mathcal{M})$. Lemma 2.2 implies that $\mu(p + q) = \mu(p) + \mu(q)$ whenever $pq = 0$ in $\mathcal{P}(\mathcal{M})$, that is, μ is finitely additive. Furthermore, by Lemma 2.1(g), $T(p)$ is a projection in B for every $p \in \mathcal{P}(\mathcal{M})$. Therefore,

$$\|\mu(p)\| = \|T(p)\| \leq 1,$$

for every $p \in \mathcal{P}(\mathcal{M})$.

Therefore the above mapping μ is a bounded B -valued finitely additive measure on $\mathcal{P}(\mathcal{M})$. By the Bunce-Wright-Mackey-Gleason theorem (cf. [6, Theorem A] or [7]), there exists a bounded linear operator $G : \mathcal{M} \rightarrow B$ satisfying

$$(2.2) \quad T(p) = \mu(p) = G(p),$$

for every $p \in \mathcal{P}(\mathcal{M})$. Let us consider an algebraic element $b = \sum_{i=1}^n \lambda_i p_i$ in \mathcal{M} , where $\lambda_i \in \mathcal{C}$ and p_1, \dots, p_n are mutually orthogonal projections in \mathcal{M} . Lemma 2.3(b) and (2.2) assure that

$$T(b) = \sum_{i=1}^n \lambda_i T(p_i) = \sum_{i=1}^n \lambda_i G(p_i) = G(b),$$

for every algebraic element b in \mathcal{M} .

Since every self-adjoint element in a von Neumann algebra can be approximated in norm by algebraic elements, it follows from the continuity of T and G that $T(a) = G(a)$ for every $a = a^*$ in \mathcal{M} and consequently

$$(2.3) \quad T(a + b) = G(a + b) = G(a) + G(b) = T(a) + T(b)$$

for every $a, b \in \mathcal{M}_{sa}$, that is, $T|_{\mathcal{M}_{sa}} : \mathcal{M}_{sa} \rightarrow B$ is linear. Lemma 2.4 implies that T is linear. Finally [31, Corollary 3.6] or [30, Theorem 3.9], imply that the mapping T is a $*$ -homomorphism. \square

The case of von Neumann algebras containing a Type I_2 direct summand must be treated independently.

Lemma 2.6. *Let \mathcal{M} be a von Neumann algebra factor, B a C^* -algebra, and let $T : \mathcal{M} \rightarrow B$ be a (not necessarily linear) 2-local $*$ -homomorphism. Then the following statements hold:*

- (a) *If there exists $a \in \mathcal{M}$ with $a \neq 0$ and $T(a) = 0$ then $T = 0$;*
- (b) *If $T \neq 0$ then T is a 2-local $*$ -monomorphism, that is, for every $a, b \in \mathcal{M}$ there exists a $*$ -monomorphism $\Phi_{a,b}$ satisfying $T(a) = \Phi_{a,b}(a)$ and $T(b) = \Phi_{a,b}(b)$;*
- (c) *If $T \neq 0$ then T is an isometry, that is, $\|T(a)\| = \|a\|$, for every $a \in \mathcal{M}$.*

Proof. Since the kernel of every $*$ -homomorphism $\pi : \mathcal{M} \rightarrow B$ is a weak*-closed ideal of \mathcal{M} , we can easily see that a every non-zero $*$ -homomorphism $\pi : \mathcal{M} \rightarrow B$ is a $*$ -monomorphism and an isometry.

(a) Suppose $T(a) = 0$, for an element $a \in \mathcal{M} \setminus \{0\}$. For each b in \mathcal{M} , take a $*$ -homomorphism $\Phi_{a,b}$ satisfying $0 = T(a) = \Phi_{a,b}(a)$ and $T(b) = \Phi_{a,b}(b)$. It follows from the above that $\Phi_{a,b} = 0$ and hence $T(b) = 0$. The statements (b) and (c) are clear from the above. \square

Proposition 2.7. *Let B be a C^* -algebra and let $T : M_2(\mathbb{C}) \rightarrow B$ be a (not necessarily linear) 2-local $*$ -homomorphism. Then T is linear and a $*$ -homomorphism.*

Proof. To simplify notation, let us write $e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $e_2 = e_1^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $p_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and $p_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. By Lemma 2.1, $T(a^2) = T(a)^2$, $T(a^*) = T(a)^*$ and $T(a)T(a)^* = T(aa^*)$, and $T(a)^*T(a) = T(a^*a)$, for every $a \in M_2(\mathbb{C})$. To simplify notation we set $z = \lambda p_1 + \mu e_1 + \alpha e_2 + \beta p_2$. Considering the $*$ -homomorphisms $\Phi_{e_1,z}$ we have

$$\begin{aligned}
 \lambda T(p_1) + \mu T(e_1) + \alpha T(e_2) + \beta T(p_2) &= \lambda T(e_1 e_1^*) + \mu T(e_1) + \alpha T(e_1^*) + \beta T(e_1^* e_1) \\
 &= \lambda T(e_1) T(e_1)^* + \mu T(e_1) + \alpha T(e_1)^* + \beta T(e_1)^* T(e_1) \\
 &= \lambda \Phi_{e_1,z}(e_1) \Phi_{e_1,z}(e_1)^* + \mu \Phi_{e_1,z}(e_1) + \alpha \Phi_{e_1,z}(e_1)^* + \beta \Phi_{e_1,z}(e_1)^* \Phi_{e_1,z}(e_1) \\
 &= \Phi_{e_1,z}(\lambda e_1 e_1^* + \mu e_1 + \alpha e_1^* + \beta e_1^* e_1) \\
 &= \Phi_{e_1,z}(\lambda p_1 + \mu e_1 + \alpha e_2 + \beta p_2) = T(\lambda p_1 + \mu e_1 + \alpha e_2 + \beta p_2).
 \end{aligned}$$

Since $\{p_1, e_1, e_2, p_2\}$ is a basis of $M_2(\mathbb{C})$, the above identity shows that T is linear. The proof concludes by [30, Theorem 3.9]. \square

We shall establish now a strengthened version of Lemma 2.3 for 2-local $*$ -homomorphisms. We recall that an element e in a C^* -algebra A is said to be a *partial isometry* when ee^* (equivalently, e^*e) is a projection. For each partial isometry $e \in A$, the elements ee^* and e^*e are called the left and right support projections of e , respectively.

Lemma 2.8. *Let $T : A \rightarrow B$ be a (not necessarily linear) 2-local $*$ -homomorphism between C^* -algebras. Let e_1, \dots, e_n be a family of mutually orthogonal partial isometries in A . Then*

$$T \left(\sum_{i=1}^n \lambda_i e_i \right) = \sum_{i=1}^n \lambda_i T(e_i),$$

for every $\lambda_1, \dots, \lambda_n \in \mathbb{C}$.

Proof. Let us note that by Lemma 2.1(i), and since $\sum_{i=1}^n e_i$ is a partial isometry, $T(e_i)$, and $T \left(\sum_{i=1}^n e_i \right)$ are partial isometries in B . Lemma 2.1(h) assures that $T(e_i) \perp T(e_j)$, for every $i \neq j$, and hence $\sum_{i=1}^n T(e_i)$ also is a partial isometry in B (cf. Lemma 2.1(i)).

Take $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ and set $z = \sum_{i=1}^n \lambda_i e_i$. The identity

$$\begin{aligned} T \left(\sum_{i=1}^n \lambda_i e_i \right) T(e_j^* e_j) &= T \left(\sum_{i=1}^n \lambda_i e_i \right) T(e_j)^* T(e_j) \\ &= \Phi_{z, e_j} \left(\sum_{i=1}^n \lambda_i e_i \right) \Phi_{z, e_j}(e_j)^* \Phi_{z, e_j}(e_j) = \lambda_j \Phi_{z, e_j}(e_j e_j^* e_j) \\ &= \lambda_j \Phi_{z, e_j}(e_j) = \lambda_j T(e_j), \end{aligned}$$

is valid for every j . Since $e_1^* e_1, \dots, e_n^* e_n$ are mutually orthogonal projections in A , Lemma 2.2 implies that

$$T \left(\sum_{j=1}^n e_j^* e_j \right) = \sum_{j=1}^n T(e_j^* e_j).$$

Set $p = \sum_{j=1}^n e_j^* e_j$ and consider the $*$ -homomorphism $\Phi_{z, p}$. It follows from the above that

$$\begin{aligned} T \left(\sum_{i=1}^n \lambda_i e_i \right) &= T \left(\left(\sum_{i=1}^n \lambda_i e_i \right) \left(\sum_{j=1}^n e_j^* e_j \right) \right) \\ &= \Phi_{z, p} \left(\left(\sum_{i=1}^n \lambda_i e_i \right) \left(\sum_{j=1}^n e_j^* e_j \right) \right) = \Phi_{z, p} \left(\sum_{i=1}^n \lambda_i e_i \right) \Phi_{z, p} \left(\sum_{j=1}^n e_j^* e_j \right) \\ &= T \left(\sum_{i=1}^n \lambda_i e_i \right) T \left(\sum_{j=1}^n e_j^* e_j \right) = T \left(\sum_{i=1}^n \lambda_i e_i \right) \left(\sum_{j=1}^n T(e_j^* e_j) \right) \end{aligned}$$

$$= \sum_{j=1}^n T \left(\sum_{i=1}^n \lambda_i e_i \right) T(e_j^* e_j) = \sum_{j=1}^n \lambda_j T(e_j).$$

□

Let a be an element in a von Neumann algebra \mathcal{M} . Following the notation in [32, §1.10], the least projection p in \mathcal{M} such that $ap = a$ (respectively, $pa = a$) is called the *right support projection* (respectively, the *left support projection*) of a and is denoted by $r(a)$ (respectively, $l(a)$). If a is self-adjoint, $l(a) = r(a)$ is simply called the *support* projection of a and is denoted by $s(a)$.

We recall at this point that a mapping f from a C^* -algebra A into a Banach space B is said to be *orthogonally additive* if for every a, b in A with $a \perp b$, we have $f(a + b) = f(a) + f(b)$.

Proposition 2.9. *Let \mathcal{M} be a von Neumann algebra, let B be a C^* -algebra, and Let $T : \mathcal{M} \rightarrow B$ be a (not necessarily linear) 2-local $*$ -homomorphism. Then T is orthogonally additive.*

Proof. Let a and b be two orthogonal elements in \mathcal{M} . We consider the polar decompositions of a and b in the form $a = u_a |a|$ and $b = u_b |b|$, where $|a| = (a^* a)^{\frac{1}{2}}$, $|b| = (b^* b)^{\frac{1}{2}}$, u_a and u_b are partial isometries in \mathcal{M} with $u_a^* u_a = s(|a|)$ and $u_b^* u_b = s(|b|)$ (cf. [32, §1.12]). Since $a \perp b$ we have $|a| \perp |b|$, $u_a \perp u_b$ and $|a + b| = |a| + |b|$.

Let $\mathcal{M}_{|a|}$, $\mathcal{M}_{|b|}$ and $\mathcal{M}_{\{|a|, |b|\}}$ denote the von Neumann subalgebras of \mathcal{M} generated by $|a|$, $|b|$ and $\{|a|, |b|\}$, respectively. Since $\mathcal{M}_{|a|}$ and $\mathcal{M}_{|b|}$ are abelian von Neumann algebras and $\mathcal{M}_{\{|a|, |b|\}}$ coincides with the orthogonal sum $\mathcal{M}_{|a|} \oplus^\infty \mathcal{M}_{|b|}$, we deduce that $\mathcal{M}_{\{|a|, |b|\}}$ is an abelian von Neumann algebra. We define a bounded linear operator $\Psi : \mathcal{M}_{|a|} \oplus^\infty \mathcal{M}_{|b|} \rightarrow \mathcal{M}$ given by

$$\Psi(x + y) := u_a x + u_b y, \quad (x \in \mathcal{M}_{|a|}, y \in \mathcal{M}_{|b|}).$$

We consider the mapping $T \circ \Psi : \mathcal{M}_{|a|} \oplus^\infty \mathcal{M}_{|b|} \rightarrow B$. We observe that, for each projection p in $\mathcal{M}_{|a|} \oplus^\infty \mathcal{M}_{|b|}$, $\Psi(p)$ is a partial isometry in \mathcal{M} , and hence $T \circ \Psi(p)$ also is a partial isometry in B (compare Lemma 2.1(i) or the first part in the proof of Lemma 2.8), which gives $\|T \circ \Psi(p)\| \leq 1$, for every projection p in $\mathcal{M}_{|a|} \oplus^\infty \mathcal{M}_{|b|}$.

Let p_1, \dots, p_n be mutually orthogonal projections in $\mathcal{M}_{|a|} \oplus^\infty \mathcal{M}_{|b|}$. Since $\Psi(p_1), \dots, \Psi(p_n)$ are mutually orthogonal partial isometries in \mathcal{M} , Lemma 2.8 proves that

$$T \circ \Psi \left(\sum_{i=1}^n p_i \right) = T \left(\sum_{i=1}^n \Psi(p_i) \right) = \sum_{i=1}^n T(\Psi(p_i)).$$

That is, $T \circ \Psi : \mathcal{P}(\mathcal{M}_{|a|} \oplus^\infty \mathcal{M}_{|b|}) \rightarrow B$, $p \mapsto T \circ \Psi(p)$, is a bounded B -valued finitely additive measure on the set $\mathcal{P}(\mathcal{M}_{|a|} \oplus^\infty \mathcal{M}_{|b|})$ of all projections in $\mathcal{M}_{|a|} \oplus^\infty \mathcal{M}_{|b|}$. Since $\mathcal{M}_{|a|} \oplus^\infty \mathcal{M}_{|b|}$ is an abelian von Neumann algebra, it

follows from the Bunce-Wright-Mackey-Gleason theorem (cf. [6, Theorem A] or [7]), that there exists a bounded linear operator $G : \mathcal{M}_{|a|} \oplus^\infty \mathcal{M}_{|b|} \rightarrow B$ satisfying

$$T \circ \Psi(p) = G(p),$$

for every $p \in \mathcal{P}(\mathcal{M}_{|a|} \oplus^\infty \mathcal{M}_{|b|})$. The continuity argument applied in the proof of Theorem 2.5 shows that

$$T \circ \Psi(z) = G(z),$$

for every $z = z^*$ in $\mathcal{M}_{|a|} \oplus^\infty \mathcal{M}_{|b|}$, and then

$$T \circ \Psi(z_1 + z_2) = G(z_1 + z_2) = G(z_1) + G(z_2) = T \circ \Psi(z_1) + T \circ \Psi(z_2),$$

for every $z_1, z_2 \in (\mathcal{M}_{|a|} \oplus^\infty \mathcal{M}_{|b|})_{sa}$. Taking $z_1 = |a|$ and $z_2 = |b|$ we obtain $T(a + b) = T(a) + T(b)$. \square

By a simple induction argument, combined with Proposition 2.9, we get:

Corollary 2.10. *Let $(\mathcal{M}_i)_{i=1}^n$ be a finite family of von Neumann algebras and let B be a C^* -algebra. Suppose that, for every i , every 2-local $*$ -homomorphism $T : \mathcal{M}_i \rightarrow B$ is linear. Then every 2-local $*$ -homomorphism*

$$T : \bigoplus_{i=1, \dots, n}^{\ell_\infty} \mathcal{M}_i \rightarrow B \text{ is linear.} \quad \square$$

Corollary 2.11. *Every (not necessarily linear) 2-local $*$ -homomorphism from a Type I_2 von Neumann algebra into a C^* -algebra is linear and a $*$ -homomorphism.*

Proof. Let \mathcal{M} be a Type I_2 von Neumann algebra and let $T : \mathcal{M} \rightarrow B$ be a 2-local $*$ -homomorphism from \mathcal{M} into a C^* -algebra. By standard classification theory of von Neumann algebras (see, for example, [32, Theorem 2.3.3]) we may suppose that

$$\mathcal{M} = C(K) \otimes M_2(\mathbb{C}),$$

where $C(K)$ is the algebra of all continuous functions on a compact Stonean space K .

Let p_1, \dots, p_m be mutually orthogonal projections in $C(K)$ with $p_1 + \dots + p_m = 1$. The von Neumann subalgebra

$$\mathcal{M}_{p_1, \dots, p_m} = p_1 \otimes M_2(\mathbb{C}) \oplus \dots \oplus p_m \otimes M_2(\mathbb{C})$$

is C^* -isomorphic to the ℓ_∞ -sum $\bigoplus_{i=1, \dots, m}^{\ell_\infty} M_2(\mathbb{C})$. Since the restricted mapping

$T|_{\mathcal{M}_{p_1, \dots, p_m}} : \mathcal{M}_{p_1, \dots, p_m} \rightarrow B$ is a 2-local $*$ -homomorphism, we deduce, via Proposition 2.7 and Corollary 2.10, that $T|_{\mathcal{M}_{p_1, \dots, p_m}}$ is linear. Fix $x, y \in \mathcal{M}$. By standard arguments (compare [28, Lemma 8.3]), for each $\varepsilon > 0$, there exist a subalgebra of the form $\mathcal{M}_{p_1, \dots, p_m}$, $x_\varepsilon, y_\varepsilon \in \mathcal{M}_{p_1, \dots, p_m}$ such that $\|x - x_\varepsilon\| < \frac{\varepsilon}{4}$, and $\|y - y_\varepsilon\| < \frac{\varepsilon}{4}$. Then, by Lemma 2.1(e),

$$\|T(x + y) - T(x) - T(y)\| \leq \|T(x + y) - T(x_\varepsilon + y_\varepsilon)\| + \|T(x_\varepsilon) - T(x)\|$$

$$+ \|T(y_\varepsilon) - T(y)\| < \|(x + y) - (x_\varepsilon + y_\varepsilon)\| + \|x_\varepsilon - x\| + \|y_\varepsilon - y\| < \varepsilon.$$

Since ε was arbitrarily chosen, we get $T(x + y) = T(x) + T(y)$. \square

The main result of this section is a consequence of Theorem 2.5, Corollary 2.11 and Corollary 2.10.

Theorem 2.12. *Every (not necessarily linear) 2-local $*$ -homomorphism from a von Neumann algebra into a C^* -algebra is linear and a $*$ -homomorphism.*

Proof. Let \mathcal{M} be a von Neumann algebra and let T be a 2-local $*$ -homomorphism from \mathcal{M} into a C^* -algebra B . By [32, Proposition 2.2.10 and Theorem 2.3.2], \mathcal{M} decomposes as the ℓ_∞ -sum of two von Neumann algebras \mathcal{M}_1 and \mathcal{M}_2 , where \mathcal{M}_1 contains no Type I_2 direct summand and \mathcal{M}_2 is a Type I_2 von Neumann algebra. Theorem 2.5 and Corollary 2.11, $T|_{\mathcal{M}_1}$ and $T|_{\mathcal{M}_2}$ are linear. The linearity of T follows from Corollary 2.10. \square

We can rediscover now some of the results commented at the introduction.

Corollary 2.13. *Every (not necessarily linear) surjective 2-local $*$ -automorphism on a von Neumann algebra is a $*$ -automorphism.* \square

3. 2-LOCAL $*$ -HOMOMORPHISMS ON DUAL C^* -ALGEBRAS

A projection p in a C^* -algebra A is said to be *minimal* if $pAp = \mathbb{C}p$. A partial isometry e in A is said to be *minimal* if ee^* (equivalently, e^*e) is a minimal projection. The *socle* of A , $\text{soc}(A)$, is defined as the linear span of all minimal projections in A . The *ideal of compact elements* in A , $K(A)$, is defined as the norm closure of $\text{soc}(A)$. A C^* -algebra is said to be *dual* or *compact* if $A = K(A)$. We refer to [22, §2], [3] and [36] for the basic references on dual C^* -algebras.

Following standard notation, for each hermitian element h in a C^* -algebra A , the symbol A_h will denote the C^* -subalgebra of A generated by h .

Lemma 3.1. *Let $T : A \rightarrow B$ be a (not necessarily linear) 2-local $*$ -homomorphism between C^* -algebras. Then, for each $h \in A_{sa}$, $T|_{A_h} : A_h \rightarrow B$ is a linear mapping.*

Proof. Consider an element $b \in A_h$ of the form $b = \sum_{k=1}^m \alpha_k h^k$ and the $*$ -homomorphism $\Phi_{h,b}$. In this case,

$$T(b) = \Phi_{h,b} \left(\sum_{k=1}^m \alpha_k h^k \right) = \sum_{k=1}^m \alpha_k \Phi_{h,b}(h)^k = \sum_{k=1}^m \alpha_k T(h)^k,$$

which proves that T is linear on the linear span of the set $\{a^k : k \in \mathbb{N}\}$. We conclude by continuity that $T|_{A_h}$ is linear. \square

Let A be a C^* -algebra. Following the notation in [1], a *positive quasi-linear functional* is a function $\rho : A \rightarrow \mathbb{C}$ such that

- (i) ρ_{A_h} is a positive linear functional for each $h \in A_{sa}$;
- (ii) $\rho(a + ib) = \rho(a) + i\rho(b)$, for every $a, b \in A_{sa}$.

When the mapping ρ also satisfies that $\sup\{\rho(a) : a \in A, \|a\| \leq 1, a \geq 0\} = 1$, then ρ is called a *quasi-state* on A .

Let $T : A \rightarrow B$ be a (not necessarily linear) 2-local $*$ -homomorphism between C^* -algebras. For each positive functional $\phi \in B_+^*$, Lemmas 2.4 and 3.1 assure that $\phi \circ T : A \rightarrow \mathbb{C}$ is a positive multiple of a quasi-state on A . When A coincides with the C^* -algebra $K(H)$ of all compact operators on a complex Hilbert space H with $\dim(H) \geq 3$, Corollary 2 in [1] implies that $\phi \circ T$ is linear. Having in mind that for each self-adjoint element $h \in B$ we have $\|h\| = \sup\{\phi(h) : \phi \geq 0, \|\phi\| = 1\}$ (cf. [32, Proposition 1.5.4]), we deduce that T is linear. Combining these arguments with Proposition 2.7 we get:

Proposition 3.2. *Let H be a complex Hilbert space, B a C^* -algebra, and let $T : K(H) \rightarrow B$ be a (not necessarily linear) 2-local $*$ -homomorphism. Then T is linear and a $*$ -homomorphism. \square*

By [3, Theorem 8.2.], we know that every compact or dual C^* -algebra A decomposes as a c_0 -sum of the form $A = \left(\bigoplus_{\lambda} K(H_{\lambda}) \right)_{c_0}$, where each H_{λ} is a complex Hilbert space. Suppose B is a C^* -algebra and

$$T : \left(\bigoplus_{\lambda} K(H_{\lambda}) \right)_{c_0} \rightarrow B$$

is a (not necessarily linear) 2-local $*$ -homomorphism. Proposition 3.2 implies that $T|_{K(H_{\lambda})} : K(H_{\lambda}) \rightarrow B$ is a linear homomorphism for every λ . For each pair of elements $a, b \in A$ and $\varepsilon > 0$ we can find a natural m , $\lambda_1, \dots, \lambda_m$,

finite dimensional subspaces $\tilde{H}_{\lambda_i} \subseteq H_{\lambda_i}$ and elements $a_{\varepsilon}, b_{\varepsilon} \in \bigoplus_{i=1, \dots, m}^{\ell_{\infty}} K(\tilde{H}_{\lambda_i})$

satisfying $\|a - a_{\varepsilon}\| < \frac{\varepsilon}{4}$, and $\|b - b_{\varepsilon}\| < \frac{\varepsilon}{4}$. Set $A_1 := \bigoplus_{i=1, \dots, m}^{\ell_{\infty}} K(\tilde{H}_{\lambda_i})$. By

Corollary 2.10, the mapping $T|_{A_1} : A_1 \rightarrow B$ is linear. Therefore, by Lemma 2.1(e),

$$\begin{aligned} \|T(a + b) - T(a) - T(b)\| &\leq \|T(a + b) - T(a_{\varepsilon} + b_{\varepsilon})\| + \|T(a_{\varepsilon}) - T(a)\| \\ &\quad + \|T(b_{\varepsilon}) - T(b)\| \leq \|(a + b) - (a_{\varepsilon} + b_{\varepsilon})\| + \|a_{\varepsilon} - a\| + \|b_{\varepsilon} - b\| < \varepsilon. \end{aligned}$$

The arbitrariness of $\varepsilon > 0$ proves the additivity of T . The final statement follows from [30, Theorem 3.9].

Theorem 3.3. *Let A be a dual or compact C^* -algebra, B a C^* -algebra, and let $T : A \rightarrow B$ be a (not necessarily linear) 2-local $*$ -homomorphism. Then T is linear and a $*$ -homomorphism. \square*

Corollary 3.4. *Every (not necessarily linear) surjective 2-local $*$ -homomorphism T on a prime C^* -algebra A , with $T(K(A)) \neq \{0\}$ is a $*$ -homomorphism. Consequently, every (not necessarily linear) surjective 2-local $*$ -automorphism on a prime C^* -algebra with non-zero socle is a $*$ -automorphism.*

Proof. By Theorem 3.3, $T|_{K(A)} : K(A) \rightarrow B$ is linear and a $*$ -homomorphism. Therefore, $\ker(T|_{K(A)})$ is a closed ideal of A . We recall that every prime C^* -algebra with non-zero socle is primitive and hence its socle is contained in every non-zero closed ideal of A . Thus, $\ker(T|_{K(A)}) = \{0\}$.

Let us take $a, b \in A$, $x \in K(A)$. By the hypothesis of 2-locality and Theorem 3.3,

$$\begin{aligned} T(x)T(a+b)T(x) &= T(x(a+b)x) = T(xax) + T(xbx) \\ &= T(x)T(a)T(x) + T(x)T(b)T(x). \end{aligned}$$

Since T is surjective, we can find $z \in A$ such that $T(z) = T(a+b) - T(a) - T(b)$. It follows from the above that $T(xzx) = 0$ and hence $xzx = 0$, for every $x \in K(A)$. The essentiality of $K(A)$ proves that $z = 0$, witnessing that $T(a+b) = T(a) + T(b)$. \square

Remark 3.5. In [24, Theorem 6], the authors prove that for every unital prime C^* -algebra A , which has a nontrivial idempotent or whose unit element is properly infinite, every surjective 2-local $*$ -automorphism, $T : A \rightarrow A$, is a $*$ -automorphism. It should be remarked here that the result is true for every C^* -algebra A without requiring any additional hypothesis. Indeed, as noted in [24], T is a surjective isometry, which, by the Mazur-Ulam theorem, implies that T is linear. In particular, T is a linear local $*$ -homomorphism, and hence a $*$ -homomorphism by [30, Theorem 3.9].

4. 2-LOCAL JORDAN $*$ -HOMOMORPHISMS ON JBW^* -ALGEBRAS

A JB^* -algebra or a *Jordan C^* -algebra* is a complex Jordan Banach algebra \mathcal{J} equipped with an algebra involution $*$ satisfying $\|U_a(a^*)\| = \|a\|^3$, $a \in \mathcal{J}$, where $U_a(x) := 2(a \circ x) \circ a - a^2 \circ x$. The self-adjoint part \mathcal{J}_{sa} of a JB^* -algebra \mathcal{J} is a JB -algebra in the usual sense employed in [10, 18]. The reciprocal statement also holds, each JB algebra is the self-adjoint part of a unique JB^* -algebra [35]. A JBW^* -algebra is a JB^* -algebra which is also a dual Banach space. For the standard definitions and properties of JB^* - and JBW^* -algebras we refer to [18, 10] and [35].

Let $\mathcal{J}_1, \mathcal{J}_2$ be Jordan Banach algebras. A (not necessarily linear nor continuous) mapping $T : \mathcal{J}_1 \rightarrow \mathcal{J}_2$ is said to be a *2-local Jordan homomorphism* if for every $a, b \in \mathcal{J}_1$ there exists a bounded (linear) Jordan homomorphism $\Phi_{a,b} : \mathcal{J}_1 \rightarrow \mathcal{J}_2$, depending on a and b , such that $\Phi_{a,b}(a) = T(a)$ and $\Phi_{a,b}(b) = T(b)$. When \mathcal{J}_1 and \mathcal{J}_2 are JB^* -algebras, a mapping $T : \mathcal{J}_1 \rightarrow \mathcal{J}_2$ is called a *2-local Jordan $*$ -homomorphism* if for every $a, b \in \mathcal{J}_1$ there exists a Jordan $*$ -homomorphism $\Phi_{a,b} : \mathcal{J}_1 \rightarrow \mathcal{J}_2$, depending on a and b , such that $\Phi_{a,b}(a) = T(a)$ and $\Phi_{a,b}(b) = T(b)$.

The statements (a), (b) and (d) – (h) in Lemma 2.1 remain valid for 2-local Jordan homomorphisms between Jordan Banach algebras and for 2-local Jordan $*$ -homomorphisms between JB $*$ -algebras, with the particularity that elements a and b in a JB $*$ -algebra \mathcal{J} are *orthogonal* ($a \perp b$) if and only if $(a \circ b^*) \circ x + (x \circ b^*) \circ a - (a \circ x) \circ b^* = 0$ for every $x \in \mathcal{J}$, or equivalently $(a \circ a^*) \circ b + (b \circ a^*) \circ a - (a \circ b) \circ a^* = 0$ (see [8, Lemma 1.1]). Clearly, when a is a self-adjoint element $a \perp b$ if and only if $a^2 \circ b = 0$. Further, if a is positive, $a \perp b$ if and only if $a \circ b = 0$ (cf. [9, Lemma 4.1]).

Let $T : \mathcal{J}_1 \rightarrow \mathcal{J}_2$ be a (not necessarily linear) 2-local Jordan $*$ -homomorphism between JB $*$ -algebras and let p_1, \dots, p_n be a family of mutually orthogonal projections in \mathcal{J}_1 . The arguments in the proofs of Lemmas 2.2 and 2.3 can be slightly modified to be valid in the Jordan setting in order to prove that

$$(4.1) \quad T \left(\sum_{i=1}^n \lambda_i p_i \right) = \sum_{i=1}^n \lambda_i T(p_i),$$

for every $\lambda_1, \dots, \lambda_n \in \mathbb{C}$. Furthermore, Lemmas 2.4 and 3.1 are true for 2-local Jordan $*$ -homomorphisms between JB $*$ -algebras, and thus

$$(4.i) \quad T(a + ib) = T(a) + iT(b), \text{ for every } a, b \in (\mathcal{J}_1)_{sa};$$

$$(4.ii) \quad \text{For each } h \in (\mathcal{J}_1)_{sa}, T_{(\mathcal{J}_1)_h} : (\mathcal{J}_1)_h \rightarrow \mathcal{J}_2 \text{ is a linear mapping, where } (\mathcal{J}_1)_h \text{ denotes the JB}^*\text{-subalgebra generated by the element } h.$$

When in the proof of Theorem 2.5 we replace the Bunce-Wright-Mackey-Gleason theorem for von Neumann algebras (cf. [6, Theorem A] or [7]) with an appropriate version for JBW-algebras (see [5, Theorem 2.1]) we obtain the following:

Theorem 4.1. *Let \mathcal{J} be a JBW $*$ -algebra without Type I_2 part and let \mathcal{B} be a JB $*$ -algebra. Suppose $T : \mathcal{J} \rightarrow \mathcal{B}$ is a (not necessarily linear) 2-local Jordan $*$ -homomorphism. Then T is linear and a Jordan $*$ -homomorphism.* \square

We recall now a result which is part of the folklore in C $*$ -algebra theory: suppose p is a projection in a unital C $*$ -algebra A and x is a norm-one (self-adjoint) element in A satisfying that $pxp = p$ then

$$(4.2) \quad x = p + (1 - p)x(1 - p).$$

Indeed, since $1 \geq \|px\|$ and

$$px = p + px(1 - p),$$

it follows that

$$(px)(px)^* = p + px(1 - p)x^*p,$$

is a positive norm-one element in pAp . Moreover, since $\|(px)(px)^*\| \leq 1$, and the element $px(1 - p)x^*p$ also is positive in pAp , it must be zero, and hence $px(1 - p) = 0$. We similarly get $(1 - p)xp = 0$ and the desired statement (4.2).

Now, let p be a projection in a unital JB*-algebra \mathcal{J} and let x be a norm-one self-adjoint element in \mathcal{J} satisfying that $U_p(x) = p$ then

$$(4.3) \quad x = p + U_{(1-p)}(x).$$

In effect, by the Shirshov-Cohn theorem [18, 2.4.14], the JB*-subalgebra \mathcal{B} of \mathcal{J} generated by p , x and the unit element is special, that is, there exists a C*-algebra A such that \mathcal{B} is a JB*-subalgebra of A . In such a case, p is a projection in \mathcal{B} , and hence in A , and it is satisfied that $\|x\| = 1$, and $pxp = U_p(x) = p$. We deduce from (4.2) that $x = p + (1-p)x(1-p) = p + U_{(1-p)}(x)$.

A stronger version of (4.2) and (4.3) was established by Friedman and Russo in [14, Lemma 1.6] for tripotents in a JB*-triple.

To deal with JBW*-algebras of Type I_2 , we shall need the following partial result.

Proposition 4.2. *Let \mathcal{J} a JB-algebra, and let $T : M_2(\mathbb{R})_{sa} \rightarrow \mathcal{J}$ be a (not necessarily linear) 2-local Jordan homomorphism. Then T is linear and a Jordan homomorphism.*

Proof. Replacing \mathcal{J} with \mathcal{J}^{**} we can always assume that \mathcal{J} is unital. By Lemma 2.6, whose statement remains valid for JBW-algebra factors, we may assume that T is a 2-local Jordan monomorphism.

Let us denote $u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $p_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and $p_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Clearly, $T(p_1)$ and $T(p_2)$ are mutually orthogonal projections, $T(u)^3 = T(u)$, $T(p_i) \circ T(u) = \frac{1}{2}T(u)$ (and hence, $U_{T(p_1)}(T(u)) = 0 = U_{1-T(p_1)}(T(u))$), and $T(u)^2 = T(1)$.

First we prove that

$$(4.4) \quad T(\lambda p_1 + \mu u) = \lambda T(p_1) + \mu T(u),$$

for every $\lambda, \mu \in \mathbb{R}$. For the proof we assume $\lambda, \mu \neq 0$. Indeed, set $z = \lambda p_1 + \mu u$ and consider the Jordan homomorphisms $\Phi_{z,p_1}, \Phi_{u,z} : M_2(\mathbb{R})_{sa} \rightarrow \mathcal{J}$. The equalities

$$T(z) = \lambda \Phi_{z,p_1}(p_1) + \mu \Phi_{z,p_1}(u) = \lambda T(p_1) + \mu \Phi_{z,p_1}(u),$$

$$T(z) = \lambda \Phi_{z,u}(p_1) + \mu \Phi_{z,u}(u) = \lambda \Phi_{z,u}(p_1) + \mu T(u),$$

imply that

$$\Phi_{z,u}(p_1) = T(p_1) + \frac{\mu}{\lambda}(\Phi_{z,p_1}(u) - T(u)).$$

The elements $\Phi_{z,u}(p_1)$ and $T(p_1)$ are norm-one projections. Since $U_{p_1}(u) = 0 = U_{1-p_1}(u)$,

$$U_{T(p_1)}(T(u)) = 0 = U_{1-T(p_1)}(T(u)),$$

and

$$\begin{aligned} U_{T(p_1)}(\Phi_{z,p_1}(u)) &= U_{\Phi_{z,p_1}(p_1)}(\Phi_{z,p_1}(u)) = 0, \\ 0 &= U_{1-\Phi_{z,p_1}(p_1)}(\Phi_{z,p_1}(u)) = U_{1-T(p_1)}(\Phi_{z,p_1}(u)), \end{aligned}$$

we deduce from (4.3) that

$$\frac{\mu}{\lambda}(\Phi_{z,p_1}(u) - T(u)) = 0,$$

which gives $\Phi_{z,p_1}(u) = T(u)$, and hence $T(\lambda p_1 + \mu u) = \lambda T(p_1) + \mu T(u)$.

Similarly, we show that

$$(4.5) \quad T(\lambda p_2 + \mu u) = \lambda T(p_2) + \mu T(u),$$

for every $\lambda, \mu \in \mathbb{R}$.

To conclude the proof we shall prove that

$$T(\lambda p_1 + \mu u + \gamma p_2) = \lambda T(p_1) + \mu T(u) + \gamma T(p_2),$$

for every $\lambda, \mu, \gamma \in \mathbb{R}$. Pick $\lambda, \mu, \gamma \in \mathbb{R} \setminus \{0\}$ and set $z = \lambda p_1 + \mu u + \gamma p_2$. Applying (4.4) we get

$$(4.6) \quad \begin{aligned} T(z) &= \Phi_{z,\lambda p_1 + \mu u}(z) = \Phi_{z,\lambda p_1 + \mu u}(\lambda p_1 + \mu u) + \gamma \Phi_{z,\lambda p_1 + \mu u}(p_2) \\ &= T(\lambda p_1 + \mu u) + \gamma \Phi_{z,\lambda p_1 + \mu u}(p_2) = \lambda T(p_1) + \mu T(u) + \gamma \Phi_{z,\lambda p_1 + \mu u}(p_2). \end{aligned}$$

And on the other hand,

$$(4.7) \quad T(z) = \Phi_{z,p_2}(z) = \lambda \Phi_{z,p_2}(p_1) + \mu \Phi_{z,p_2}(u) + \gamma T(p_2).$$

Combining (4.6) and (4.7) we get

$$\Phi_{z,\lambda p_1 + \mu u}(p_2) = T(p_2) + \frac{1}{\gamma}(\lambda \Phi_{z,p_2}(p_1) - \lambda T(p_1)) + \frac{\mu}{\gamma}(\Phi_{z,p_2}(u) - T(u)).$$

Having in mind that $\Phi_{z,\lambda p_1 + \mu u}(p_2)$ and $T(p_2)$ are norm-one projections,

$$U_{T(p_2)}(T(u)) = 0 = U_{1-T(p_2)}(T(u)),$$

$$U_{T(p_2)}(\Phi_{z,p_2}(u)) = U_{\Phi_{z,p_2}(p_2)}(\Phi_{z,p_2}(u)) = 0,$$

$$U_{1-T(p_2)}(\Phi_{z,p_2}(u)) = U_{1-\Phi_{z,p_2}(p_2)}(\Phi_{z,p_2}(u)) = 0,$$

by orthogonality,

$$U_{1-T(p_2)}(T(p_1)) = T(p_1),$$

and

$$U_{1-T(p_2)}(\Phi_{z,p_2}(p_1)) = U_{1-\Phi_{z,p_2}(p_2)}(\Phi_{z,p_2}(p_1)) = \Phi_{z,p_2}(p_1),$$

we deduce from (4.3) that

$$\Phi_{z,p_2}(u) = T(u).$$

Therefore, (4.7) writes in the form

$$(4.8) \quad T(z) = \lambda \Phi_{z,p_2}(p_1) + \mu T(u) + \gamma T(p_2).$$

Now, applying (4.5) we get

$$(4.9) \quad T(z) = \Phi_{z,\gamma p_2 + \mu u}(z) = \lambda \Phi_{z,\gamma p_2 + \mu u}(p_1) + \mu T(u) + \gamma T(p_2).$$

Independently,

$$(4.10) \quad T(z) = \Phi_{z,p_1}(z) = \lambda T(p_1) + \mu \Phi_{z,p_1}(u) + \gamma \Phi_{z,p_1}(p_2).$$

Arguing as in the previous paragraphs, we deduce from these identities that $\Phi_{z,p_1}(u) = T(u)$, and thus (4.10) writes in the form

$$(4.11) \quad T(z) = \lambda T(p_1) + \mu T(u) + \gamma \Phi_{z,p_1}(p_2).$$

Combining (4.8) and (4.11) we obtain:

$$\Phi_{z,p_1}(p_2) = T(p_2) + \frac{\lambda}{\gamma}(\Phi_{z,p_2}(p_1) - T(p_1)).$$

Since $\Phi_{z,p_2}(p_1) \perp \Phi_{z,p_2}(p_2) = T(p_2)$, $T(p_2) \perp T(p_1)$, the projection $\Phi_{z,p_1}(p_2)$ is equal to the orthogonal sum of the projection $T(p_2)$ and the element $\frac{\lambda}{\gamma}(\Phi_{z,p_2}(p_1) - T(p_1))$. It follows that $\frac{\lambda}{\gamma}(\Phi_{z,p_2}(p_1) - T(p_1))$ is a projection orthogonal to $T(p_2)$. However, $\Phi_{z,p_1}(p_2) \perp \Phi_{z,p_1}(p_1) = T(p_1)$ assures that

$$\frac{\lambda}{\gamma}(\Phi_{z,p_2}(p_1) - T(p_1)) = (\Phi_{z,p_1}(p_2) - T(p_2)) \perp T(p_1).$$

That is,

$$(4.12) \quad (\Phi_{z,p_2}(p_1) - T(p_1)) \perp (T(p_2) + T(p_1)) = T(p_1 + p_2) = T(1)$$

(compare (4.1)).

It is known that for each $a \in M_2(\mathbb{R})_{sa}$, $U_{T(1)}T(a) = U_{\Phi_{a,1}(1)}\Phi_{a,1}(a) = \Phi_{a,1}U_1(a) = T(a)$. In particular, the identity (4.8) implies that

$$U_{T(1)}(\Phi_{z,p_2}(p_1)) = \Phi_{z,p_2}(p_1),$$

and consequently

$$U_{T(1)}(\Phi_{z,p_2}(p_1) - T(p_1)) = \Phi_{z,p_2}(p_1) - T(p_1).$$

This identity together with (4.12) show that $\Phi_{z,p_2}(p_1) - T(p_1) = 0$. Finally, the identity (4.8) writes in the form

$$T(\lambda p_1 + \mu u + \gamma p_2) = \lambda T(p_1) + \mu T(u) + \gamma T(p_2),$$

which proves the linearity of T . \square

Let \mathcal{J}_1 and \mathcal{J}_2 be two JBW-algebras satisfying that, for each JB-algebra \mathcal{B} , every 2-local Jordan homomorphism $S : \mathcal{J}_i \rightarrow \mathcal{B}$ is linear. Suppose $T : \mathcal{J}_1 \oplus^\infty \mathcal{J}_2 \rightarrow \mathcal{B}$ is a 2-local Jordan homomorphism. Given $a \in \mathcal{J}_1$, $b \in \mathcal{J}_2$, and $\varepsilon > 0$, we can find two families p_1, \dots, p_m and q_1, \dots, q_k of mutually orthogonal projections in \mathcal{J}_1 and \mathcal{J}_2 , respectively, and real numbers λ_i, μ_j such that

$$\left\| a - \sum_{i=1}^m \lambda_i p_i \right\| < \frac{\varepsilon}{4}, \text{ and } \left\| b - \sum_{j=1}^k \mu_j q_j \right\| < \frac{\varepsilon}{4}.$$

We note that $p_i \perp q_j$, for every i, j . In this case, by (4.1),

$$\|T(a+b) - T(a) - T(b)\| \leq \left\| T(a+b) - T\left(\sum_{i=1}^m \lambda_i p_i + \sum_{j=1}^k \mu_j q_j\right) \right\|$$

$$+ \left\| T \left(\sum_{i=1}^n \lambda_i p_i \right) - T(a) \right\| + \left\| T \left(\sum_{j=1}^m \mu_j q_j \right) - T(b) \right\| < \varepsilon.$$

Since ε was arbitrarily chosen, the assumptions on \mathcal{J}_1 and \mathcal{J}_2 show that T is linear.

A induction argument shows:

Corollary 4.3. *Let $(\mathcal{J}_i)_{i=1}^n$ be a finite family of JBW-algebras and let \mathcal{B} be a JB-algebra. Suppose that, for every i , every 2-local Jordan homomorphism $T : \mathcal{J}_i \rightarrow \mathcal{B}$ is linear. Then every 2-local Jordan homomorphism $T : \bigoplus_{i=1, \dots, n}^{\ell_\infty} \mathcal{J}_i \rightarrow \mathcal{B}$ is linear.* \square

We can establish now the Jordan version of Corollary 2.11.

Corollary 4.4. *Every (not necessarily linear) 2-local Jordan $*$ -homomorphism from a Type I_2 JBW $*$ -algebra into a JB $*$ -algebra is linear and a Jordan $*$ -homomorphism.*

Proof. Having in mind (4.i), we observe that it is enough to prove that $T : \mathcal{J}_{sa} \rightarrow \mathcal{B}_{sa}$ is additive. We follow some of the arguments given by Bunce and Hamhalter in [4, Theorem in page 158]. Let $x, y \in \mathcal{J}_{sa}$. As noted by the authors in [4, proof of Theorem in page 158], the JBW-subalgebra, $\mathcal{J}_{sa}(x, y)$, of \mathcal{J}_{sa} generated by x and y identifies, via [34, Theorem 2],

$$\mathcal{J}_{sa}(x, y) = C(K, \mathbb{R}) \otimes M_2(\mathbb{R})_{sa},$$

where $C(K, \mathbb{R})$ is the algebra of all continuous real-valued functions on a compact Stonean space K .

When in the proof of Corollary 2.11, Proposition 2.7 and Corollary 2.10 are replaced with Proposition 4.2 and Corollary 4.3, respectively, the arguments remain valid to prove the desired statement. \square

Combining Theorem 4.1, Corollaries 4.4 and 4.3 and the structure theory of JBW $*$ -algebras (cf. [18, §5]) we obtain the following Jordan version of Theorem 2.12.

Theorem 4.5. *Every (not necessarily linear) 2-local Jordan $*$ -homomorphism from a JBW $*$ -algebra into a JB $*$ -algebra is linear and a Jordan $*$ -homomorphism.* \square

Corollary 4.6. *Every (not necessarily linear) 2-local Jordan $*$ -homomorphism from a von Neumann algebra into a C^* -algebra is linear and a Jordan $*$ -homomorphism.* \square

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